

FORCING DISCRETIZATION AND DETERMINATION IN QUANTUM HISTORY THEORIES

Bob Coecke

Imperial College of Science, Technology & Medicine, Theoretical Physics Group,
The Blackett Laboratory, South Kensington, London SW7 2BZ;

Free University of Brussels (VUB), Department of Mathematics (FUND),
Pleinlaan 2, B-1050 Brussels, Belgium (permanent mailing address);
e-mail: bocoecke@vub.ac.be.

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Abstract

We present a formally deterministic representation for quantum history theories where we obtain the probabilistic structure via a discrete contextual variable: no continuous probabilities are as such involved at the primal level — we conceive as a history theory any theory that deals with sequential quantum measurements but remains essentially a dichotomic propositional theory. A major part of the paper consists of a concise survey of arXiv: quant-ph/0008061 and quant-ph/0008062.

1 INTRODUCTION

In this paper we propose and study a model for history theories in which the probability structure emerges from a finite number of contextual happenings, any next happening having a fixed chance to occur under the condition that the previous one happened. Although this model cannot have a canonical mathematical status since it has been proved that this type of representation in general admits no essentially unique “smallest one” [8]ⁱⁱ, it provides insight in the emergence of logicity in the “History Projection Operator” setting [14], and it illustrates how deterministic behavior can be encoded beyond those interpretations of quantum history theories that are interpretationally restricted by so-called consistency or quasi-consistency (e.g., approximate decoherence). The particular motivation for this “paradigm case study” finds its origin in structural considerations towards a theory of quantum gravity [4, 15, 19]. As argued in [16], although the relative frequency interpretation of probability justifies the continuous interval as the codomain for value assignment, “... in the quantum gravity regime standard ideas of space and time might break down

in such a way that the idea of spatial or temporal ‘ensembles’ is inappropriate. For the other main interpretations of probability — subjective, logical, or propensity — there seems to be no compelling *a priori* reason why probabilities should be real numbers.” Our model should be envisioned as a deconstructive step unraveling the probabilistic continuum as it appears in standard quantum theory, reducing it explicitly to a discrete temporal sequence of (contextual) events. The as such emerging temporal sequence is then easier to manipulate towards alternative encoding of contextual events, e.g., in propositional terms. It also enables a separate treatment of internal (the system’s) and external (the context’s) time-encoding variable.

Although quantum history theories are currently most frequently envisioned in a context of so-called decoherence we prefer to take the minimal perspective that a history theory is a theory that deals with sequential quantum measurements but remains essentially a dichotomic propositional theory. This is formally encoded in a rigid way in the History Projection Operator-approach [14]. We also mention recently studied sequential structures in the context of quantum logic, of which references can be found in [10], resulting in a dynamic disjunctive quantum logic, which provides an appropriate formal context to discuss the logicity of history theories.

A general theory on deterministic contextual models can be found in [8]. Note here that what we consider as contextuality is that in a measurement there is an interaction between the system and its context and that precisely this interaction to some extent may influence the outcome of a measurement. A lack of knowledge on the precise interaction then yields quantum-type uncertainties [1]. Besides this interpretational issue, classical representations are important since we think classical, so even without giving any conceptual significance to the representation, it provides a mode to think deterministically in terms of determined trajectories of the system’s state, without having to reconcile with concrete non-canonical constructs like pilot-wave mechanics.

2 OUTCOME DETERMINATION VIA CONTEXTUAL MODELS

We will present the required results in full abstraction such that the reader clearly sees which structural ingredient of quantum theory determines existence of contextual models. For details and proofs we refer to [8]. Let $\mathcal{B}(\mathbb{R}^\nu)$ denote the Borel subsets of \mathbb{R}^ν .

Definition 1. A ‘probabilistic measurement system’ is given by:

- (i) A set of states Σ and a set of measurements \mathcal{E} ;
- (ii) For each $e \in \mathcal{E}$ an outcome set $O_e \in \mathcal{B}(\mathbb{R}^\nu)$, a σ -field $\mathcal{B}(O_e)$ of O_e -subsets and (Kolmogorovian) probability measures $P_{p,e} : \mathcal{B}(O_e) \rightarrow [0, 1]$ for each $p \in \Sigma$.

The canonical example is that of quantum theory with every Hilbert space ray ψ representing a state, every self-adjoint operator H representing a measurement with its spectrum $O_H \subseteq \mathbb{R}$ as outcome set where the σ -structure $\mathcal{B}(O_H)$ is inherited from that of $\mathcal{B}(\mathbb{R})$ and with probability measures $P_{\psi,H}(E) := \langle \psi | P_E \psi \rangle$ where P_E denotes the spectral projector for $E \in \mathcal{B}(O_H)$. In benefit of insight and also for notational convenience we will from now on assume that the measurements $e \in \mathcal{E}$ are represented in a one to one way by their outcome sets O_e — note that whenever \mathcal{E} can be represented by points of $\mathbb{R}^{\nu'}$ it then suffices to consider $\mathbb{R}^{\nu} \times \mathbb{R}^{\nu'} = \mathbb{R}^{\nu+\nu'}$ in stead of \mathbb{R}^{ν} to fulfill this assumption, taking $O_e \times \{e\}$ as the corresponding outcome set. We stress however that the results listed below also hold in absence of this assumption [8]^{i,ii}.

Definition 2. A ‘pre-probabilistic hidden measurement system’ is given by:

- (i) A set of states Σ and a set of measurements \mathcal{E} ;
- (ii) Sets $\mathcal{O} \subseteq \mathcal{B}(\mathbb{R}^{\nu})$ and Λ that parameterize \mathcal{E} , i.e., $\mathcal{E} = \{e_{\lambda,O} | \lambda \in \Lambda, O \in \mathcal{O}\}$, and each $e \in \mathcal{E}$ goes equipped with a map $\varphi_{\lambda,O} : \Sigma \rightarrow O$.

We can represent $\{\varphi_{\lambda,O} | \lambda \in \Lambda\}$ as $\varphi_O : \Sigma \times \Lambda \rightarrow O : (p, \lambda) \mapsto \varphi_{\lambda,O}(p)$ giving Λ a similar formal status as the set of states Σ , or as $\Delta\Lambda_O : \Sigma \times \mathcal{B}(O) \rightarrow \mathcal{P}(\Sigma) : (p, E) \mapsto \{\lambda | \varphi_O(p, \lambda) \in E\}$ where $\mathcal{P}(\Lambda)$ denotes the set of subsets of Λ . The core of this definition is that given a state $p \in \Sigma$ and a value $\lambda \in \Lambda$ we have a completely determined outcome $\varphi_O(p, \lambda)$. These pre-probabilistic hidden measurement systems encode as such fully deterministic settings.

Definition 3. Whenever for a given pre-probabilistic hidden measurement system $(\Sigma, \mathcal{E}(\mathcal{O}, \Lambda), \{\varphi_O\}_{O \in \mathcal{O}})$ there exists a σ -field $\mathcal{B}(\Lambda)$ of Λ -subsets that satisfies $\bigcup_{O \in \mathcal{O}} \{\Delta\Lambda_O(p, E) | (p, E) \in \Sigma \times \mathcal{B}(O)\} \subseteq \mathcal{B}(\Lambda)$, it defines a ‘probabilistic hidden measurement system’ if a probability measure $\mu : \mathcal{B}(\Lambda) \rightarrow [0, 1]$ is also specified.

The condition on $\Delta\Lambda$ requires that all $\Delta\Lambda_O(p, E)$ are $\mathcal{B}(\Lambda)$ -measurable, such that to all triples (p, O, E) we can assign a value $P_{p,O}(E) := \mu(\Delta\Lambda_O(p, E)) \in [0, 1]$. As such, any probabilistic hidden measurement system defines a measurement system. The question then rises whether every probabilistic measurement system (MS) can be encoded as a probabilistic hidden measurement system (HMS). The answer to this question is yes [8]ⁱ, §4.2, Theorem 1,2 & 3: *There always exists a canonical HMS-representation for $\Lambda \cong [0, 1]$, $\mathcal{B}(\Lambda) \cong \mathcal{B}([0, 1])$ (i.e., the Borel sets in $[0, 1]$) and $\mu_u([0, a]) := a$, i.e., uniformly distributed* — the proof goes via a construction using the Loomis-Sikorski Theorem [17, 20] and Marczewski’s Lemma [13]. It makes as such sense to investigate how the different possible HMS-representations for different non-isomorphic pairs $(\mathcal{B}(\Lambda), \mu)$ are structured — below it will become clear what we mean here by non-isomorphic. First we will discuss an example that illustrates the above; it traces back to [1] and details and illustrations can be found in [2, 8]ⁱⁱ. Consider the states of a spin- $\frac{1}{2}$ entity encoded as a point on the Poincaré sphere $\Sigma_o(\cong \mathbb{C}^2/\mathbb{C}) \subseteq \mathbb{R}^3$. Then any pair of antipodically located points of Σ_o encodes

mutual orthogonal states, as such encodes mutual orthogonal one-dimensional projectors and thus a (dichotomic) measurement. Let $p \in \Sigma_\circ$, let $(\alpha, \neg\alpha)$ be a pair of mutual orthogonal points of Σ_\circ and let Λ be the diagonal connecting α and $\neg\alpha$. Let $x_p \in \Lambda$ be the orthogonal projection of p on the diagonal Λ . Then, for $\lambda \in [x_p, \neg\alpha]$, i.e., $x_p \in [\alpha, \lambda]$, we set $\varphi(p, \lambda) = \alpha$ and for $\lambda \in [\alpha, x_p[$, i.e., $x_p \in]\lambda, \neg\alpha]$, we set $\varphi(p, \lambda) = \neg\alpha$. One then verifies that for $\mu_\circ := \mathcal{B}([\alpha, \neg\alpha]) \rightarrow [0, 1] : [\alpha, (1-x)\alpha + x\neg\alpha] \mapsto x$, i.e., uniformly distributed, we obtain exactly the probability structure for spin- $\frac{1}{2}$ in quantum theory.¹ An interpretational proposal of this model could be the following:[1, 2, 3] *Rather than decomposing states as in so-called hidden variable theories, here we decompose the measurements in deterministic ones — the probability measure μ should then be envisioned as encoding the lack of knowledge on the interaction of the measured system with its environment, including measurement device.*

We now introduce a notion of “relative size” of HMS-representations, justifying the use of “smaller”. Given a σ -algebra² and probability measure $\mu : \mathcal{B} \rightarrow [0, 1]$ denote by \mathcal{B}/μ the σ -algebra of equivalence classes $[E]$ with respect to the relation $E \sim E' \Leftrightarrow \mu(E' \cap E^c) = \mu(E \cap E'^c) = 0$, i.e., iff E and E' coincide up to a symmetric difference of measure zero. The ordering of \mathcal{B}/μ is inherited from \mathcal{B} . For notational convenience denote the induced measure $\mathcal{B}/\mu \rightarrow [0, 1] : [E] \mapsto \mu(E)$ again by μ . Given two pairs (\mathcal{B}, μ) and (\mathcal{B}', μ') consisting of separable σ -algebras and probability measures on them set:

$$\bullet (\mathcal{B}, \mu) \leq (\mathcal{B}', \mu') \Leftrightarrow \begin{cases} \exists f : \mathcal{B}/\mu \rightarrow \mathcal{B}'/\mu', \text{ an injective } \sigma\text{-morphism} \\ \mu' \circ f = \mu \end{cases}$$

We call (\mathcal{B}, μ) and (\mathcal{B}', μ') equivalent, denoted $(\mathcal{B}, \mu) \sim (\mathcal{B}', \mu')$, whenever in the above f is a σ -isomorphism. Given two MS (Σ, \mathcal{E}) and (Σ', \mathcal{E}') we set:

$$\bullet (\Sigma, \mathcal{E}) \sim_{MS} (\Sigma', \mathcal{E}') \Leftrightarrow \begin{cases} \exists s : \Sigma \rightarrow \Sigma', \exists t : \mathcal{E} \rightarrow \mathcal{E}', \text{ both bijections} \\ \forall e \in \mathcal{E}, \exists f_e : \mathcal{B}(O_e) \rightarrow \mathcal{B}(O_{t(e)}), \text{ a } \sigma\text{-isomorphism} \\ \forall p \in \Sigma, \forall e \in \mathcal{E} : P_{s(p), t(e)} \circ f_e = P_{p, e} \end{cases}$$

Via this equivalence relation we can define a relation \leq_{MS} between classes of measurement systems \mathcal{M} and \mathcal{M}' as $\mathcal{M} \leq_{MS} \mathcal{M}'$ if for all $(\Sigma, \mathcal{E}) \in \mathcal{M}$ there exists $(\Sigma', \mathcal{E}') \in \mathcal{M}'$ such that $(\Sigma, \mathcal{E}) \sim_{MS} (\Sigma', \mathcal{E}')$, i.e., if \mathcal{M} is included in \mathcal{M}' up to MS-equivalence. We can then prove the following:

(i) $(\mathcal{B}, \mu) \sim (\mathcal{B}', \mu')$ if and only if $(\mathcal{B}, \mu) \leq (\mathcal{B}', \mu')$ and $(\mathcal{B}', \mu') \leq (\mathcal{B}, \mu)$ — [8]ⁱⁱ, §3, Lemma 1; thus, the equivalence classes with respect to \sim constitute a partially ordered set (poset) for the ordering induced by \leq ; we will denote the

¹As shown in [6, 9] this deterministic model for spin- $\frac{1}{2}$ in \mathbb{R}^3 can be generalized to \mathbb{R}^3 -models for arbitrary spin- $N/2$. The states are then represented in the so called Majorana representation [18, 5], i.e., as N copies of Σ_\circ . Correct probabilistic behavior is then obtained by introducing entanglement between the N different “spin- $\frac{1}{2}$ systems”.

²I.e., a “pointless” σ -field. In particular, it follows from the Loomis-Sikorski theorem [17, 20] that all separable σ -algebras (i.e., which contain a countable dense subset) can be represented as a σ -field — it as such also follows that assuming that $\mathcal{B}(\Lambda)$ is a σ -field and not an abstracted σ -algebra imposes no formal restriction.

set of these equivalence classes by \mathbf{M} , a class in it will be denoted via a member of it as $[\mathcal{B}, \mu]$.

(ii) When setting $\mathbf{M}_{HMS} := \{\mathcal{M}[\mathcal{B}(\Lambda), \mu] \mid [\mathcal{B}(\Lambda), \mu] \in \mathbf{M}\}$ where $\mathcal{M}[\mathcal{B}(\Lambda), \mu]$ stands for all HMS with $\mathcal{B}(\Lambda')$ and μ' such that $(\mathcal{B}'(\Lambda'), \mu') \in [\mathcal{B}(\Lambda), \mu]$, we have that $(\mathcal{B}(\Lambda), \mu) \leq (\mathcal{B}'(\Lambda'), \mu')$ and $\mathcal{M}[\mathcal{B}(\Lambda), \mu] \leq_{MS} \mathcal{M}[\mathcal{B}'(\Lambda'), \mu']$ are equivalent [8]ⁱⁱ, §3, Theorem 2. This then results in:

Theorem 1. (\mathbf{M}, \leq) and $(\mathbf{M}_{HMS}, \leq_{MS})$ are isomorphic posets.

One of the crucial ingredients in (ii) above and also in the proof for general existence with $\Lambda \cong [0, 1]$ is the following: when setting $\Delta\mathbf{M}(\Sigma, \mathcal{E}) := \{(\mathcal{B}(O_e), P_{p,e}) \mid p \in \Sigma, e \in \mathcal{E}\}$, we obtain that Σ, \mathcal{E} admits a HMS-representation with $\mathcal{B}(\Lambda)$ and μ if and only if $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \leq (\mathcal{B}(\Lambda), \mu)$, where the order applies pointwisely to the elements of $\Delta\mathbf{M}(\Sigma, \mathcal{E})$ [8]ⁱ, §4.2, Theorem 1. Using this and Theorem 1 above we can now translate properties of \mathbf{M} to propositions on the existence of certain HMS-representations. We obtain the following:

(i) (\mathbf{M}, \leq) is not a join-semilattice, thus: *In general there exists no smallest HMS-representation.* As such we will have to refine our study to particular settings where we are able to make statements whether there exists a smallest one, and if not, whether we can say at least something on the cardinality of Λ .

(ii) One can prove a number of criteria on $\Delta\mathbf{M}(\Sigma, \mathcal{E})$ that force $(\mathcal{B}(\Lambda), \mu) \sim (\mathcal{B}([0, 1]), \mu_u)$ as such assuring existence of a smallest representation. Among these the following. Let $\mathbf{M}_{finite} := \{(\mathcal{B}(X), \mu) \in \mathbf{M} \mid X \text{ is finite}\}$. *If $\mathbf{M}_{finite} \subseteq \Delta\mathbf{M}(\Sigma, \mathcal{E})$ then Λ cannot be discrete.* It then follows for example that quantum theory restricted to measurements with a finite number of outcomes still requires $\Lambda \cong [0, 1]$.

(iii) Let $\mathbf{M}_N := \{(\mathcal{B}(X), \mu) \in \mathbf{M} \mid X \text{ has at most } N \text{ elements}\}$. *If $\Delta\mathbf{M}(\Sigma, \mathcal{E}) \subseteq \mathbf{M}_N$ then there exists a HMS-representation with $\Lambda \cong \mathbb{N}$.* Thus, quantum theory restricted to those measurements with at most a fixed number N of outcomes has discrete HMS-representation.

(iv) *If $\Delta\mathbf{M}(\Sigma, \mathcal{E}) = \mathbf{M}_N$ then there exists no smallest HMS-representation. Neither does it exist when fixing the number of outcomes.* So there is no essentially unique smallest HMS-representation for N -outcome quantum theory.

Although there exists no smallest and as such no canonical discrete HMS-representation we will give the construction of one solution for dichotomic (or propositional) quantum theory, i.e., $N = 2$, since this will constitute the core of the model presented in this paper. We will follow [8]ⁱⁱⁱ, to which we also refer for a construction for arbitrary N . Let us denote the quantum mechanical probability to obtain a positive outcome in a measurement of a proposition or question α on a system in state p as $P_p(\alpha)$ — the outcome set consists here of “we obtain a positive answer for the question α ”, slightly abusively denoted as α itself, and “we obtain a negative answer for the question α ”, denoted as $\neg\alpha$.

Set inductively for $\lambda \in \mathbb{N}$:³

$$\bullet \varphi_\alpha(p, \lambda) := \begin{cases} \alpha & \text{iff } P_p(\alpha) \geq \frac{1}{2^\lambda} + \sum_{i=1}^{\lambda-1} \frac{\delta(\varphi_\alpha(p, i), \alpha)}{2^i} \\ \neg\alpha & \text{otherwise} \end{cases}$$

One verifies that for $\mu(\lambda) := \frac{1}{2^\lambda}$ we obtain the correct probabilities in the resulting HMS-model. This provides a discrete alternative for the above discussed \mathbb{R}^3 -model for $\text{spin-}\frac{1}{2}$. The model, including the projection x_p remains the same although we don't consider $[\alpha, \neg\alpha]$ as Λ anymore. Let $\lambda \in \Lambda' := \mathbb{N}$. Set $x_n^\lambda := (1 - \frac{n}{2^\lambda})\alpha + (\frac{n}{2^\lambda})\neg\alpha$ for $n \in \mathbb{Z}_{2^\lambda-1}$. For $x_p \in [\alpha, x_1^\lambda] \cup [x_2^\lambda, x_3^\lambda] \cup [x_4^\lambda, x_5^\lambda] \cup \dots \cup [x_{2^\lambda-1}^\lambda, \neg\alpha]$ we set $\varphi'_\alpha(p, \lambda) = \alpha$, and $\varphi'_\alpha(p, \lambda) = \neg\alpha$ otherwise. Then, for $\mu'_o := \mathcal{B}(\mathbb{N}) \rightarrow [0, 1] : \{\lambda\} \mapsto \frac{1}{2^\lambda}$ we obtain again quantum probability. Geometrically, this means that the values of $\lambda \in \Lambda$, as compared to the first model where they represents points on the diagonal, i.e., a continuous interval, or, again equivalently, decompositions of an interval in two intervals, we now consider decompositions of an interval in 2^λ equally long parts, of which there are only a discrete number of possibilities. We refer to [8]ⁱⁱ for details and illustrations concerning.

3 UNITARY, ORTHO- AND PROJECTIVE STRUCTURE

In the above discussed \mathbb{R}^3 models, rotational symmetries were implicit in their spatial geometry. However, in general the decompositions of measurements over $\mu : \mathcal{B}(\Lambda) \rightarrow [0, 1]$ go measurement by measurement so additional structure, if there is any, has to be put in by hand. It is probably fair to say that these contextual models only become non-trivial and useful when encoding physical symmetries within the maps φ_α in an appropriate manner. For sake of the argument we will distinguish between three types of symmetries that can be encoded, namely unitary, ortho- and projective ones.

i. Unitary symmetries: When considering quantum measurements with discrete non-degenerated spectrum we can represent the outcomes $\{o_i\}_i$ by the corresponding “eigenstates” $\{p_i\}_i$ via spectral decomposition, i.e., there exists an injective map $\mathcal{B}(O_e) \rightarrow \mathcal{P}(\Sigma)$ for each $e \in \mathcal{E}$. Then, specification of $\varphi : \Sigma \times \Lambda \rightarrow \{p_i\}_i$ and μ for one measurement $e_0 \in \mathcal{E}$ fixes it for any other $e \in \mathcal{E}$ by symmetry: $\varphi_e = (U \circ \varphi \circ U^{-1}) : \Lambda \times \Sigma \rightarrow \{p_{e,i}\}_i$, where $U : \Sigma \rightarrow \Sigma$ is the unitary transformation that satisfies $U(p_i) = p_{e,i}$, and $\mu_e = \mu$. This is exactly the symmetry encoded in the above described \mathbb{R}^3 -models. Note in particular that in this perspective the pairs $(\alpha, \neg\alpha)$ and $(\neg\alpha, \neg(\neg\alpha))$ should not be envisioned as merely a change of names of the outcomes, but truly as putting the

³We agree on $\mathbb{N} := \{1, 2, \dots\}$. Note here that already by non-uniqueness of binary decomposition — $\frac{1}{2} = \frac{1}{2^1} = \sum_{i \in \mathbb{N}} \frac{1}{2^{i+1}}$ — it follows that the construction below is not canonical. Obviously, there are also less pathological differences between the different non-comparable discrete representations [8]ⁱⁱ.

measurement device (or at least its detecting part) upside down.⁴ In this setting where we represent outcomes as states, the assignment of an outcome can now be envisioned as a true change of state $f_{e,\lambda} : \Sigma \rightarrow \Sigma (\supseteq O_e) : p \mapsto \varphi_e(p, \lambda)$, as such allowing to describe the behavior of the system under concatenated measurements.

ii. Projective symmetries: For non-degenerated quantum measurements, the outcomes require representation by higher dimensional subspaces so identification in terms of states now requires an injective map $\mathcal{B}(O_e) \rightarrow \mathcal{P}(\mathcal{P}(\Sigma))$. The behavior of states of the system under concatenated measurements then requires specification of a family of “projectors” $\{\pi_T : \Sigma \rightarrow T \mid T \in O_e\}$, e.g., the orthogonal projectors $\pi_A : \Sigma \rightarrow A : p \mapsto A \wedge (p \vee A^\perp)$ on the corresponding subspace A in quantum theory. The above discussed non-degenerated case fits also in this picture by setting $O_e \subseteq \{\{p\} \mid p \in \Sigma\}$ where now each $\pi_{\{p\}} : \Sigma \rightarrow \{p\}$ is uniquely determined (having a singleton codomain).

iii. Orthosymmetries: The existence of an orthocomplementation on the lattice of closed subspaces of a Hilbert space provides a dichotomic representation for measurements which can be envisioned as a pair consisting of a (to be verified) proposition α and its negation $\neg\alpha$, in quantum theory yielding $\pi_{\neg A} : \Sigma \rightarrow A^\perp : p \mapsto A^\perp \wedge (p \vee A)$. In terms of linear operator calculus we have $\pi_{\neg A} = 1 - \pi_A$, both of them being orthogonal projectors.

4 REPRESENTING QUANTUM HISTORY THEORY

Although quantum history theory involves sequential measurements, one of its goals is to remain an essentially dichotomic propositional theory. This is formally encoded in a rigid way in the “History Projection Operator”-approach [14]. The key idea here is that the form of logicity aimed at in [14] represents faithfully in the Hilbert space tensor product.⁵ Let $\mathcal{A} := (\alpha_{t_i})_i$ be a (so-called homogeneous) quantum history proposition with temporal support (t_1, t_2, \dots, t_n) . Then, rather than representing this as a sequence of subspaces $(A_i)_i$ or projectors $(\pi_i)_i$ we will either represent \mathcal{A} as a pure tensor $\otimes_i A_i$ in the lattice of closed subspaces of the tensor product of the corresponding Hilbert spaces or as the orthogonal projector $\otimes_i \pi_i$ on this subspace. The crucial property of this representation is then that $\neg\mathcal{A}$ again encodes as a projector namely

⁴The attentive reader will note that it is at this point that we escape the so-called hidden variable no-go theorems. They arise when trying to impose contextual symmetries within the states of the system by requiring that values of observables are independent of the chosen context, e.g., the proof of the Kochen-Specker theorem. Our newly introduced variable $\lambda \in \Lambda$ follows contextual manipulations in an obvious manner.

⁵At this point we mention that in the study of sequential phenomena in the axiomatic quantum theory perspective on quantum logic, sequentiality and compoundness both turn out to be specifications of a universal causal duality [10], as such providing a metaphysical perspective on the use of tensor products both for the description of compound physical systems and sequential processes.

$id - \otimes_i \pi_i$ [14], clarifying the notations $\pi_{\mathcal{A}}$ and $\pi_{\neg \mathcal{A}}$. Moreover, if $\{\mathcal{A}^i\}_i$ is a set of so-called disjoint history propositions, i.e., $\otimes_k A_k^i \perp \otimes_k A_k^j$ for $i \neq j$, then, the history proposition that expresses the disjunction of $\{\mathcal{A}^i\}_i$ sensu [14] is exactly encoded as the projector $\sum_i \otimes_k \pi_k^i$. We get as such a kind of logical setting that is still encoded in terms of projectors. Note that $\pi_{\neg \mathcal{A}}$ is not of the form $\otimes_i \pi_i$ but of the form $\sum_i \otimes_k \pi_k^i$ breaking the structural symmetry between a proposition and its negation in ordinary quantum theory.

We will now transcribe the observations in the two previous sections to this setting in order to provide a contextual deterministic model for quantum history theory with discretely originating probabilities. One could say that we will apply a split picture in terms of Schrödinger-Eisenbergh, namely we assume that on the level of unitary evolution we apply the Eisenbergh picture such that we can fix notation without reference to this evolution, but for changes of state due to measurement we will (obviously) express this in the state space. When encoding outcomes in terms of states we need to consider n copies of Σ , encoding the trajectories due to the measurements. In view of the considerations made above it will be no surprise that we will consider these trajectories as of the form $\otimes_i p_i$ in the tensor product $\otimes_i \Sigma_i$. This will require the introduction of the following “pseudo-projector”:

$$\bullet \pi_{\mathcal{A}}^{\otimes} : \Sigma \rightarrow \otimes_i \Sigma_i : p \mapsto p_{\mathcal{A}}^{\otimes} := p \otimes \pi_1(p) \otimes \dots \otimes (\pi_{n-1} \circ \dots \circ \pi_1)(p).$$

Setting $\Sigma_{\mathcal{A}}^{\otimes} := \pi_{\mathcal{A}}^{\otimes}[\Sigma] = \{p_{\mathcal{A}}^{\otimes} | p \in \Sigma\}$ then $\pi_{\mathcal{A}}^{\otimes} : \Sigma \rightarrow \Sigma_{\mathcal{A}}^{\otimes}$ encodes a bijective representation of Σ . Noting that $P_p(\mathcal{A}) := \langle p_{\mathcal{A}}^{\otimes} | \pi_{\mathcal{A}} p_{\mathcal{A}}^{\otimes} \rangle$ is the probability given by quantum theory to obtain \mathcal{A} , we then set inductively for fixed $\lambda \in \mathbb{N}$ that $\varphi_{\mathcal{A}}(p, \lambda) = \mathcal{A}$ if and only if:

$$\bullet \langle p_{\mathcal{A}}^{\otimes} | \pi_{\mathcal{A}} p_{\mathcal{A}}^{\otimes} \rangle \geq \frac{1}{2^{\lambda}} + \sum_{i=1}^{\lambda-1} \frac{\delta(\varphi_{\mathcal{A}}(p, i))}{2^i}$$

and $\varphi_{\mathcal{A}}(p, \lambda) = \neg \mathcal{A}$ otherwise. The outcome trajectories in case we obtain \mathcal{A} are then given in terms of initial states by $(\pi_{\mathcal{A}} \circ \pi_{\mathcal{A}}^{\otimes}) : \Sigma \rightarrow \otimes_i A_i$. The value $\lambda \in \mathbb{N}$ can be envisioned as follows. We assume it to be a number of contextual events, either real or virtual depending on one’s taste, and we assume that, given that some events already happened, the chance of a next one happening is equal to the chance that it doesn’t happen, so we actually consider a finite number of probabilistically balanced consecutive binary decisive processes where the result of the previous one determines whether we actually will perform the next one. Unitary symmetries are induced in the obvious way as tensored unitary operators $\otimes_i U_i$. This model then produces the statistical behavior of quantum history theory.

The breaking of the structural symmetry between a proposition and its negation manifestates itself in the most explicit way in the sense that when we have a determined outcome $\neg \mathcal{A}$ we don’t have a determined trajectory in our model — obviously one could build a fully deterministic model that also determines this by concatenation of individual deterministic models (one for each element in the temporal support), but we feel that this would not be in

accordance with the propositional flavor a history theory aims at. The negation $\neg A$ is indeed cognitive and not ontological with respect to the actual executed physical procedure or, in other words, the system's context, and one cannot expect an ontological model to encode this in terms of a formal duality. Explicitly, $\neg(A \otimes B)$ can be written both as $(\mathcal{H} \otimes \neg B) \oplus (\neg A \otimes B)$ and $(\neg A \otimes \mathcal{H}) \oplus (A \otimes \neg B)$ which clearly define different procedures with respect to imposed change of state due to the measurement. Even more explicitly, setting $\mathcal{HPO}(\{\mathcal{H}_k\}_k) := \{\sum_i \otimes_k A_k^i | A_k^i \in \mathcal{L}(\mathcal{H}_k), \otimes_k A_k^i \perp \otimes_k A_k^j \text{ for } i \neq j\}$ for $\mathcal{L}(\mathcal{H}_k)$ the lattice of closed subspaces of \mathcal{H}_k , the “ontologically faithful hull” of $\mathcal{HPO}(\{\mathcal{H}_k\}_k)$ consists then of all “ortho-ideals” $\mathcal{OI}(\mathcal{HPO}(\{\mathcal{H}_k\}_k)) :=$

$$\bullet \{ \downarrow [\{\otimes_k A_k^i\}_i] \mid A_k^i \in \mathcal{L}(\mathcal{H}_k), \otimes_k A_k^i \perp \otimes_k A_k^j \text{ for } i \neq j \}$$

where $\downarrow [-]$ assigns to a set of pure tensors all pure tensors in $\otimes_k \mathcal{H}_k$ that are smaller than at least one in the given set, this with respect to the ordering in $\mathcal{L}(\otimes_k \mathcal{H}_k)$ — the downset $\downarrow [-]$ construction makes $\mathcal{OI}(\mathcal{HPO}(\{\mathcal{H}_k\}_k))$ inherit the $\mathcal{L}(\otimes_k \mathcal{H}_k)$ -order as intersection. If a particular decomposition is specified as an element of $\mathcal{OI}(\mathcal{HPO}(\{\mathcal{H}_k\}_k))$, what means full specification of the physical procedure where summation over different sequences of pure tensors is now envisioned as choice of procedure, we can provide a deterministic contextual model, the choice of procedure itself becoming an additional variable. Conclusively, the HPO-setting “looses” part of the physical ontology that goes with an operational perspective on quantum theory,⁶ and as such, if we want to provide a deterministic representation for general inhomogeneous history propositions sensu the one we obtained for the homogeneous ones, we formally need to restore this part of the physical ontology, e.g. as $\mathcal{OI}(\mathcal{HPO}(\{\mathcal{H}_k\}_k))$.

5 FURTHER DISCUSSION

In this paper we didn't provide an answer and we even didn't pose a question. We just provided a new way to think about things, slightly confronting the usual consistency or decoherence perspective for history theories. Even if one does not subscribe to the underlying deterministic nature of the model it still exhibits what a minimal representation of the indeterministic ingredients can be, as such representing it in a more tangible way. With respect to the non-existence of a smallest representation, in view of other physical considerations it could be that one of the constructible discrete models presents itself as the truly canonical one, e.g., equilibrium or other thermodynamical considerations, metastatistical ones, emerging from additional modelization.

⁶A choice that is motivated by the traditional consistent history setting and its interpretation as well as by a particular semantical perspective on quantum logic as a whole.

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